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Using Adomian decomposition methods for solving systems of nonlinear partial differential equations

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Abstract: **In this paper, we apply the Adomian decomposition method (ADM) and Modified decomposition method (MDM) on two different types of nonlinear partial differential equations (PDEs), has been solved by using the homotopy perturbation method combined with new transform (NTHPM). But after solved by (MADM) we found (MADM) has less of computational work than (NTHPM), more effective, powerful and simple than (NTHPM).**

Keywords: **Systems of nonlinear partial differential equations, Adomian decomposition method, Modified decomposition method, homotopy perturbation method combined with new transform.**

I. INTRODUCTION

The system of PDEs arises in many areas of mathematics, engineering and physical sciences. These systems are too complicated to be solved exactiy so it is still very difficult to get closed form solutions for most problems. A vast class of analytical and numerical methods has been proposed to solve such problems. Such as the Adomian decomposition method (ADM) [1,2], the variational iteration method [3,4], the homotopy perturbation method (HPM) [5-7], and the differential transform method [8,9]. But many systems such as system of high dimensional equations, the required calculations to obtain it is solution in some time may be too complicated.

Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular[10-13]. The method proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral equations [13-17].

1.1 The Adomian Decomposition Method

In this section of nonlinear partial differential equations will be examined by using Adomian decomposition method. Systems of nonlinear partial differential equations arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of chemical reaction-diffusion model. To achieve our goal in handling systems of nonlinear partial differential equations , we write a system in an operator form by

$$L_t u + L_x v + N_1(u, v) = g_1$$

$$L_t u + L_x v + N_2(u, v) = g_2 \qquad (1)$$

With initial data

$$u(x, 0) = f_1(x),$$

$$v(x, 0) = f_2(x), \qquad (2)$$

Where $L_t$and L_x are considered, without loss of generality, first order partial differential operators, N_1 and N_2 are nonlinear operators, and $g_1$and g_2 are source terms.Operating with the integral operator L_t^{-1} to the system (1) and using the initial data (2) yields

$$u(x,t) = f_1(x) + L_t^{-1}g_1 - L_t^{-1}L_x v - L_t^{-1}N_1(u,v),$$

$$v(x,t) = f_2(x) + L_t^{-1}g_2 - L_t^{-1}L_x v - L_t^{-1}N_2(u,v), \qquad (3)$$

The linear unknown functions $u(x,t)$ and $v(x,t)$ can be decomposed by infinite series of components

$$u(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t),$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n\,(x,t) \qquad (4)$$

However, the nonlinear operators $N_1(u,v)$ and $N_2(u,v)$ should be represented by using the infinite series of the so-called Adomian polynomials A_n and B_n

As follows:

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n,$$

$$N_2(u,v) = \sum_{n=0}^{\infty} B_n, \qquad (5)$$

Where $u_n(x,t)$ and $v_n(x,t)$, $n \geq 0$ are the components of $u(x,t)$ and $v(x,t)$ respectively that will be recurrently determined, and A_n and B_n, $n \geq 0$ are Adomian polynomials that can be generated for all forms of nonlinearity. Substituting (4) and (5) into (3) gives

$$\sum_{n=0}^{\infty} u_n\,(x,t) = f_1(x) + L_t^{-1}g_1 - L_t^{-1}L_x(\sum_{n=0}^{\infty} v_n) - L_t^{-1}(\sum_{n=0}^{\infty} A_n),$$

$$\sum_{n=0}^{\infty} v_n\,(x,t) = f_2(x) + L_t^{-1}g_2 - L_t^{-1}L_x(\sum_{n=0}^{\infty} u_n) - L_t^{-1}(\sum_{n=0}^{\infty} B_n). \qquad (6)$$

Two recursive relations can be constructed from (1.6) given by

$$u_0(x,t) = f_1(x) + L_t^{-1}g_1,$$

$$u_{k+1}(x,t) = -L_t^{-1}(L_x v_k) - L_t^{-1}(A_k), \qquad k \geq 0, \qquad (7)$$

$$v_0(x,t) = f_2(x) + L_t^{-1}g_2,$$

$$v_{k+1}(x,t) = -L_t^{-1}(L_x u_k) - L_t^{-1}(B_k), \qquad k \geq 0, \qquad (8)$$

It is an essential feature of the decomposition method that the zeroth components

$u_0(x,t)$ and $v_0(x,t)$ are defined always by all terms that arise from initial data and from integrating the source terms. Having defined the zeroth pair (u_0, v_0), the remaining pair (u_k, v_k) $\;k \geq 1\;$ can be obtained in a recurrent manner by using (7) and (8) . Additional pairs for the decomposition series solutions normally account for higher accuracy. Having determined the components of $u(x,t)$ and (x,t) , the solution (u,v) of the system follows immediately in the form of a power series expansion upon using (4).

1.2 The Modified Decomposition Method

The modified decomposition method will further accelerate the convergence of the series solution. It is to be noted that the modified decomposition method will be applied, wherever it is appropriate, to all partial differential equations of many order. To give a clear description of the technique, we consider the partial differential equation in an operator form

$$Lu + Ru = g, \qquad (9)$$

Where L is the highest order derivative, R is a linear differential operator of less order or equal order to L, and g is the source term. Operating with the inverse operator on (9) we obtain

$$u = f - L^{-1}(Ru), \qquad (10)$$

Where f represents the terms arising from the given initial condition and form grating the source term g. Define the solution u as an infinite sum of components defined by

$$u = \sum_{n=0}^{\infty} u_n \qquad (11)$$

The aim of the decomposition method is to determine the components $u_n, n \geq 0$

Recurrently and elegantly. To achieve this goal, the decomposition method admits the use of the recursive relation

$$u_0 = f,$$

$$u_{k+1} = -L^{-1}(Ru_k), \quad k \geq 0. \qquad (12)$$

In view of (12), the components $u_n, n \geq 0$ are readily obtained.

The modified decomposition method introduces a slight variation to the recursive relation(12) that will lead to the determination of the components of u in a faster and easier way. For specific cases, the function f can be set as the sum of two partial functions, namely f_1 and f_2.In other words, we can set

$$f = f_1 + f_2 \qquad (13)$$

Using (13), we introduce a qualitative change in the formation of the recursive relation (12). To reduce the size of calculations, we identify the zeroth component u_0 by one part of f, namely f_1 or f_2. The other part of f can be added to the componentu_1 among other terms. I n other words, the modified recursive relation can be identified by

$$u_0 = f_1,$$

$$u_1 = f_2 - L^{-1}(Ru_0),$$

$$u_{k+1} = -L^{-1}(Ru_k), k \geq 1. \qquad (14)$$

Tow important remarks related to the modified method can be made here.

First, by proper selection of the functions $f_1$and f_2, the exact solution u may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the choice of $f_1$and f_2, and this can be made through trials . Second , if f consists of one term only, the standard decomposition method should be employed in this case.

1.3 Numerical Examples

Example1. Consider the following of two nonlinear equations:

$$u_t - vu_x - v_t u_y = 1 - x + y + t,$$

$$v_t - uv_x - u_t v_t = 1 - x - y - t, \qquad (15)$$

With the conditions

$$u(x, y, 0) = x + y - 1, \; v(x, y, 0) = x - y + 1. \qquad (16)$$

Solution.

Operating with L_t^{-1} we obtain

$$u(x,t) = x + y - 1 + t - tx + ty + \frac{1}{2}t^2 + L_t^{-1}(vu_x + v_t u_y),$$

$$v(x,t) = x - y + 1 + t - tx - ty - \frac{1}{2}t^2 + L_t^{-1}(uv_x + u_t v_t). \qquad (17)$$

The linear terms $u(x, y, t)$ and $v(x, y, t)$ can be represented by the decomposition series

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n\,(x,y,t),$$

$$v(x,y,t)=\sum_{n=0}^{\infty}v_n\,(x,y,t), \qquad (18)$$

And the nonlinear terms $vu_x\,, v_t u_y\,, uv_x$ and $u_t v_t$by an infinite series of polynomials $\sum_{n=0}^{\infty}A_n\,, \sum_{n=0}^{\infty}B_n\,, \sum_{n=0}^{\infty}C_n\,, \sum_{n=0}^{\infty}D_n$ respectively.

Where $A_n\,, B_n\,, C_n\,, D_n$ are the Adomian polynomials that can be generated for any form of nonlinearity. Substituting (18) and an infinite series of polynomials into (17) gives

$$\sum_{n=0}^{\infty}u_n\,(x,t)=x+y-1+t-tx+ty+\frac{1}{2}t^2+L_t^{-1}(\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}B_n),$$

$$\sum_{n=0}^{\infty}v_n\,(x,t)=x-y+1+t-tx-ty-\frac{1}{2}t^2+L_t^{-1}(\sum_{n=0}^{\infty}C_n+\sum_{n=0}^{\infty}D_n)(19)$$

To accelerate the convergence of the solution, the modified decomposition method will be applied here. The modified decomposition method defines the recursive relations in the form

$$u_0(x,y,t)=x+y-1,$$

$$u_1(x,y,t)=t-tx+ty+\frac{1}{2}t^2+L_t^{-1}(A_0+B_0),$$

$$u_{k+1}(x,y,t)=L_t^{-1}(A_k+B_k),\;\; k\geq 1. \qquad (20)$$

And

$$v_0(x,y,t)=x-y+1,$$

$$v_1(x,y,t)=t-tx-ty-\frac{1}{2}t^2+L_t^{-1}(C_0+D_0),$$

$$v_{k+1}(x,y,t)=L_t^{-1}(C_k+D_k),\;\; k\geq 1. \qquad (21)$$

The Adomian polynomials for the nonlinear term vu_x are given by

$$A_0=v_0\,u_{0_x},$$

$$A_1=v_1\,u_{0_x}+v_0\,u_{1_x},$$

$$A_2=v_2\,u_{0_x}+v_1\,u_{1_x}+v_0\,u_{2_x}$$

$$A_3=v_3\,u_{0_x}+v_2\,u_{1_x}+v_1\,u_{2_x}+v_0\,u_{3_x}$$

$$A_4=v_4\,u_{0_x}+v_3\,u_{1_x}+v_2\,u_{2_x}+v_1\,u_{3_x}+v_0\,u_{4_x}$$

For the nonlinear term $v_t u_y$ by

$$B_0=v_{0_t}u_{0_y},$$

$$B_1=v_{1_t}u_{0_y}+v_{0_t}u_{1_y},$$

$$B_2=v_{2_t}u_{0_y}+v_{1_t}u_{1_y}+v_{0_t}u_{2_y},$$

$$B_3=v_{3_t}u_{0_y}+v_{2_t}u_{1_y}+v_{1_t}u_{2_y}+v_{0_t}u_{3_y},$$

$$B_4=v_{4_t}u_{0_y}+v_{3_t}u_{1_y}+v_{2_t}u_{2_y}+v_{1_t}u_{3_y}+v_{0_t}u_{4_y},$$

For the nonlinear term uv_x by

$$C_0 = u_0\, v_{0_x},$$

$$C_1 = u_1\, v_{0_x} + u_0\, v_{1_x},$$

$$C_2 = u_2\, v_{0_x} + u_1\, v_{1_x} + u_0\, v_{2_x}$$

$$C_3 = u_3\, v_{0_x} + u_2\, v_{1_x} + u_1\, v_{2_x} + u_0\, v_{3_x}$$

$$C_4 = u_4\, v_{0_x} + u_3\, v_{1_x} + u_2\, v_{2_x} + u_1\, v_{3_x} + u_0\, v_{4_x}$$

And for the nonlinear term $u_t v_y$ by

$$D_0 = u_{0_t} v_{0_y},$$

$$D_1 = u_{1_t} v_{0_y} + u_{0_t} v_{1_y},$$

$$D_2 = u_{2_t} v_{0_y} + u_{1_t} v_{1_y} + u_{0_t} v_{2_y},$$

$$D_3 = u_{3_t} v_{0_y} + u_{2_t} v_{1_y} + u_{1_t} v_{2_y} + u_{0_t} v_{3_y},$$

$$D_4 = u_{4_t} v_{0_y} + u_{3_t} v_{1_y} + u_{2_t} v_{2_y} + u_{1_t} v_{3_y} + u_{0_t} v_{4_y},$$

Using the derived Adomian polynomials into equations (20) and (21), we obtain:

$$u_0(x,y,t) = x + y - 1, v_0(x,y,t) = x - y + 1$$

$$u_1(x,y,t) = t - tx + ty + \frac{1}{2}t^2 + L_t^{-1}(x - y + 1 + 0) = 2t + \frac{1}{2!}t^2$$

$$v_1(x,y,t) = t - tx - ty - \frac{1}{2}t^2 + L_t^{-1}(x + y - 1 + 0) = -\frac{1}{2!}t^2$$

$$u_2(x,y,t) = L_t^{-1}\left(-\frac{1}{2!}t^2 - t\right) = -\frac{1}{3!}t^3 - \frac{1}{2!}t^2$$

$$v_2(x,y,t) = L_t^{-1}\left(2t + \frac{1}{2!}t^2 - 2 - t\right) = -2t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3$$

$$u_3(x,y,t) = L_t^{-1}\left(-2t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - 2 + t + \frac{1}{2!}t^2\right) = -2t - \frac{1}{2!}t^2 + \frac{1}{3}t^3 + \frac{1}{4!}t^4$$

$$v_3(x,y,t) = -L_t^{-1}\left(t - \frac{1}{3!}t^3\right) = \frac{1}{2!}t^2 - \frac{1}{4!}t^4$$

$$u_4(x,y,t) = L_t^{-1}\left(t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \frac{1}{4!}t^4\right) = \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - \frac{1}{4!}t^4 - \frac{1}{5!}t^5$$

$$v_4(x,y,t) = L_t^{-1}\left(-t - \frac{3}{2}t^2 + 2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right)$$

$$= 2t - \frac{1}{2!}t^2 - \frac{1}{2}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5$$

$$u_5(x,y,t) = L_t^{-1}\left(2 + t - 2t^2 - \frac{1}{3}t^3 + \frac{2}{4!}t^4 + \frac{1}{5!}t^5\right) = 2t + \frac{1}{2!}t^2 - \frac{2}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{60}t^5 + \frac{1}{6!}t^6$$

$$v_5(x,y,t) = L_t^{-1}\left(-t + \frac{2}{3!}t^3 - \frac{1}{5!}t^5\right) = -\frac{1}{2!}t^2 + \frac{1}{12}t^4 - \frac{1}{6!}t^6$$

The solutions $u(x,y,t), v(x,y,t)$ in a series form are given by :

$$u = x + y - 1 + 2t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \frac{1}{2!}t^2 - 2t - \frac{1}{2!}t^2 + \frac{1}{3}t^3 + \frac{1}{4!}t^4 + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - \frac{1}{4!}t^4 - \frac{1}{5!}t^5 + 2t - \frac{1}{2!}t^2 - \frac{1}{2}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \cdots$$

$$v = x - y + 1 - \frac{1}{2!}t^2 - 2t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{2!}t^2 - \frac{1}{4!}t^4 + 2t - \frac{1}{2!}t^2 - \frac{1}{2}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 - \frac{1}{2!}t^2 + \frac{1}{12}t^4 - \frac{1}{6!}t^6 + \cdots$$

Accordingly, the solution of the system in a series form is given by

$u = x + y + t - 1$

$v = x - y - t + 1)$ (22)

Example 2. Consider the following nonlinear system of inhomogeneous partial differential equations :

$u_t - w_x v_t - \frac{1}{2} w_x u_{xx} = -4xt,$

$v_t - w_t u_{xx} = 6t,$

$w_t - u_{xx} - v_x w_t = 4xt - 2t - 2$ (23)

Subject to the initial conditions

$u(x,0) = x^2 + 1\,, \quad v(x,0) = x^2 - 1\,, w(x,0) = x^2 - 1$ (24)

Solution.

Following the analysis presented above we obtain

$u(x,t) = x^2 + 1 - 2xt^2 + L_t^{-1}(w_x v_t) + L_t^{-1}\left(\frac{1}{2} w_x u_{xx}\right),$

$v(x,t) = x^2 - 1 + 3t^2 + L_t^{-1}(w_t u_{xx}),$

$w(x,t) = x^2 - 1 + 2xt^2 - t^2 - 2t + L_t^{-1}(u_{xx}) + L_t^{-1}(v_x w_t),$ (25)

Substituting the decomposition representations for linear and nonlinear into

(25) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + 1 - 2xt^2 + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + L_t^{-1}\left(\sum_{n=0}^{\infty} \frac{1}{2} B_n\right),$$

$$\sum_{n=0}^{\infty} v_n(x,t) = x^2 - 1 + 3t^2 + L_t^{-1}\left(\sum_{n=0}^{\infty} C_n\right),$$

$$\sum_{n=0}^{\infty} w_n = x^2 - 1 + 2xt^2 - t^2 - 2t + L_t^{-1}\left(\sum_{n=0}^{\infty} u_{xx}\right) + L_t^{-1}\left(\sum_{n=0}^{\infty} D_n\right)$$

(26)

To accelerate the convergence of the solution, the modified decomposition method will be applied here. The modified decomposition method defines the recursive relations in the form

$$u_0(x,t) = x^2 + 1,$$

$$u_1(x,t) = -2xt^2 + L_t^{-1}(A_0) + L_t^{-1}\left(\frac{1}{2} B_0\right),$$

$$u_{k+1}(x,t) = L_t^{-1}\left(A_k + \frac{1}{2} B_k\right), \quad k \geq 1. \qquad (27)$$

$$v_0(x,t) = x^2 - 1,$$

$$v_1(x,t) = 3t^2 + L_t^{-1}(C_0),$$

$$v_{k+1}(x,t) = L_t^{-1}(C_k), \ \ k \geq 1. \qquad (28)$$

And

$$w_0(x,t) = x^2 - 1,$$

$$w_1(x,t) = 2xt^2 - t^2 - 2t + L_t^{-1}(u_{0_{xx}}) + L_t^{-1}(D_0),$$

$$w_{k+1}(x,t) = L_t^{-1}(u_{k_{xx}} + D_k), \ \ k \geq 1. \qquad (29)$$

For brevity, we list the first three Adomian polynomials for A_n, B_n, C_n and D_n as follows:

For $w_x v_t$ we find

$$A_0 = w_{0_x} v_{0_t},$$

$$A_1 = w_{1_x} v_{0_t} + w_{0_x} v_{1_t},$$

$$A_2 = w_{2_x} v_{0_t} + w_{1_x} v_{1_t} + w_{0_x} v_{2_t},$$

For $w_x u_{xx}$ we find

$$B_0 = w_{0_x} u_{0_{xx}},$$

$$B_1 = w_{1_x} u_{0_{xx}} + w_{0_x} u_{1_{xx}},$$

$$B_2 = w_{2_x} u_{0_{xx}} + w_{1_x} u_{1_{xx}} + w_{0_x} u_{2_{xx}},$$

For $w_t u_{xx}$ we find

$$C_0 = w_{0_t} u_{0_{xx}},$$

$$C_1 = w_{1_t} u_{0_{xx}} + w_{0_t} u_{1_{xx}},$$

$$C_2 = w_{2_t} u_{0_{xx}} + w_{1_t} u_{1_{xx}} + w_{0_t} u_{2_{xx}},$$

For $v_x w_t$ we find

$$D_0 = v_{0_x} w_{0_t},$$

$$D_1 = v_{1_x} w_{0_t} + v_{0_x} w_{1_t},$$

$$D_2 = v_{2_x} w_{0_t} + v_{1_x} w_{1_t} + v_{0_x} w_{2_t},$$

Using the derived Adomian polynomials into equations (27) , (28) and (29), we obtain:

$$u_0(x,t) = x^2 + 1, v_0(x,t) = x^2 - 1, w_0(x,t) = x^2 - 1$$

$$u_1(x,t) = -2xt^2 + L_t^{-1}(0) + L_t^{-1}(0) = -2xt^2,$$

$$v_1(x,t) = 3t^2 + L_t^{-1}(0) = 3t^2,$$

$$w_1(x,t) = 2xt^2 - t^2 - 2t + L_t^{-1}(2) + L_t^{-1}(0) = 2xt^2 - t^2,$$

$$u_2(x,t) = L_t^{-1}(12xt + 4xt - 2t) = 8xt^2 - t^2,$$

$$v_2(x,t) = L_t^{-1}(8xt - 4t) = 4xt^2 - 2t^2,$$

$$w_2(x,t) = L_t^{-1}(8x^2t - 4xt) = 4x^2t^2 - 2xt^2,$$

$$u_3(x,t) = L_t^{-1}(12t^3 + 24x^2t - 12xt) = 3t^4 + 12x^2t^2 - 6xt^2$$

$$v_3(x,t) = L_t^{-1}(16x^2t - 8xt) = 8x^2t^2 - 4xt^2,$$

$$w_3(x,t) = L_t^{-1}(16xt^3 - 8t^3 + 24t^2) = 4xt^4 - 2t^4 + 8t^3$$

The solutions $u(x,t), v(x,t)$ in a series form are given by :

$$u(x,t) = x^2 + 1 - 2xt^2 + 8xt^2 - t^2 3t^4 + 12x^2t^2 - 6xt^2 + \cdots$$

$$v(x,t) = x^2 - 1 + 4xt^2 - 2t^2 + 8x^2t^2 - 4xt^2 + \cdots$$

$$w(x,t) = x^2 - 1 + 2xt^2 - t^2 + 4x^2t^2 - 2xt^2 + 4xt^4 - 2t^4 + 8t^3 + \cdots$$

And in a closed form by:

$$u(x,t) = x^2 - t^2 + 1$$

$$v(x,t) = x^2 + t^2 - 1$$

$w(x,t) = x^2 - t^2 - 1$ (30)

Which are the exact solutions.

II. CONCLUSION

In this paper, we applied modified decomposition method (MDM) to solve systems of partial differential equations. It may be concluded that the MDM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result.

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